Finite Lovász Profiles for Coprime Affine Groups

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Abstract

We introduce the finite left and right Lovász dimensions of a class of finite groups, measuring whether the class is classified by homomorphism counts from, or into, groups of bounded order. We prove that coprime affine groups with fixed quotient have finite Lovász dimension on both sides. Fix a prime p and a finite p'-group Q. Let $\mathcal{C}_{p,Q}$ be the class of finite groups G such that

$$O_p(G) \text{ is elementary abelian}, \qquad G/O_p(G) \cong Q.$$

Equivalently, by Schur–Zassenhaus and Maschke,

$$G \cong V \rtimes Q$$

with V a semisimple $\mathbb{F}_pQ$-module, considered up to twisting by $\mathrm{Aut}(Q)$. We construct finite families of bounded-order test groups

$$\mathcal{T}_{p,Q}, \qquad \mathcal{R}_{p,Q}$$

such that, for all $G, H \in \mathcal{C}_{p,Q}$,

$$G \cong H$$

if and only if

$$|\operatorname{Hom}(T, G)| = |\operatorname{Hom}(T, H)| \quad \text{for all } T \in \mathcal{T}_{p,Q},$$

and also if and only if

$$|\operatorname{Hom}(G, R)| = |\operatorname{Hom}(H, R)| \quad \text{for all } R \in \mathcal{R}_{p,Q}.$$

Thus $\mathcal{C}_{p,Q}$ has finite left and right Lovász dimension. The construction is explicit and bounded in terms of p and Q, independently of $\dim_{\mathbb{F}_p} O_p(G)$. The left test family is built from quotients (Q/N), primitive central idempotents of $\mathbb{F}_pQ$, and Möbius inversion on the normal subgroup lattice of Q. The right test family is dual: its epimorphism counts recover surjection moments of semisimple modules. Both profiles reconstruct the semisimple $\mathbb{F}_pQ$-module $O_p(G)$ up to the natural $\mathrm{Aut}(Q)$-action. We also prove sharpness results. The class of finite abelian p-groups of exponent at most p^R has finite left Lovász dimension, classified by the cyclic tests $C_p, \ldots, C_{p^R}$. In contrast, the class of all finite abelian p-groups has infinite left Lovász dimension: no finite family of finitely generated test groups distinguishes all of them by homomorphism counts. Finally, we prove two positive cases of a recent two-object hom-count test problem. First, finite abelian groups satisfy both the left and right two-object test properties. Second, if Q is a finite p'-group with

$$\mathrm{End}(Q) = \mathrm{Aut}(Q) \cup \{0\},$$

and $V^Q = 0$, then the coprime affine groups $V \rtimes Q$ satisfy both two-object test properties. The proof reduces hom-counts to a positive-definite exponential kernel on semisimple multiplicity vectors.

Declaration of generative AI and AI-assisted technologies in the writing process

This article belongs to the Presentation theory section. Generative AI, including ChatGPT by OpenAI, was used to assist with mathematical drafting, formalization, review, and editing. The note may be preliminary, may have been only partially reviewed, and in some cases may be only AI-reviewed.

1 Introduction

Homomorphism counts are among the most basic numerical invariants of finite structures. For graphs, Lovász's theorem says that two finite graphs G, H are isomorphic if and only if

$$|\operatorname{Hom}(F, G)| = |\operatorname{Hom}(F, H)|$$

for every finite graph F. This is often called equality of the left homomorphism profile. There is also a right profile, obtained from counts

$$|\operatorname{Hom}(G, F)|.$$

This paper studies finite-group analogues of a sharper question:

When can an infinite class of finite groups be classified by homomorphism counts involving only bounded-size test

We introduce two parameters for a class $\mathcal{C}$ of finite groups:

$$\Lambda_L(\mathcal{C}), \qquad \Lambda_R(\mathcal{C}),$$

the left and right finite Lovász dimensions. Informally, $\Lambda_L(\mathcal{C}) \leq B$ means that groups in $\mathcal{C}$ are classified by the numbers

$$|\operatorname{Hom}(T, G)|$$

with $|T| \leq B$. Similarly, $\Lambda_R(\mathcal{C}) \leq B$ means that they are classified by

$$|\operatorname{Hom}(G, R)|$$

with $|R| \leq B$. The main theorem proves that both dimensions are finite for coprime affine groups with fixed quotient. Let p be a prime and Q a finite p'-group. Define

$$\mathcal{C}_{p,Q} = \{G : O_p(G) \text{ is elementary abelian and } G/O_p(G) \cong Q\}.$$

By Schur–Zassenhaus,

$$G \cong V \rtimes Q,$$

where $V = O_p(G)$ is elementary abelian. Since $p \nmid |Q|$, Maschke's theorem implies that V is a semisimple $\mathbb{F}_p Q$-module. The abstract group G is determined by V, but only up to twisting the Q-action by $\operatorname{Aut}(Q)$. Our first result is:

Theorem 1.1 (Finite two-sided Lovász dimension for coprime affine groups)**.** *For every prime p and finite p'-group Q,*

$$\Lambda_L(\mathcal{C}_{p,Q}) < \infty \qquad \textit{and} \qquad \Lambda_R(\mathcal{C}_{p,Q}) < \infty.$$

More explicitly, there exist finite families of finite groups

$$\mathcal{T}_{p,Q}, \qquad \mathcal{R}_{p,Q},$$

of order bounded only in terms of p and Q, such that for all $G, H \in \mathcal{C}_{p,Q}$,

$$G \cong H$$

if and only if

$$|\operatorname{Hom}(T, G)| = |\operatorname{Hom}(T, H)| \quad \forall T \in \mathcal{T}_{p,Q},$$

and also if and only if

$$|\operatorname{Hom}(G, R)| = |\operatorname{Hom}(H, R)| \quad \forall R \in \mathcal{R}_{p,Q}.$$

The proof of the left profile theorem is constructive. Let

$$1 = e_1 + \cdots + e_s$$

be the primitive central idempotent decomposition of $\mathbb{F}_pQ$. For every normal subgroup $N \lhd Q$, and every tuple $\mathbf{i} = (i_1, \ldots, i_d)$, we define a small semidirect product

$$T_{N,\mathbf{i}} U_{N,\mathbf{i}} \rtimes Q/N,$$

where

$$U_{N,\mathbf{i}} \bigoplus_{\ell=1}^{d} \mathbb{F}_p[Q/N]/\mathbb{F}_p[Q/N]\bar{e}_{i_\ell}.$$

Homomorphism counts from these groups into G produce sums of kernel sizes of the idempotents e_i acting on $O_p(G)$. Möbius inversion on the normal subgroup lattice of Q isolates embeddings

$$Q \hookrightarrow G.$$

The embeddings over a fixed automorphism $\alpha \in \operatorname{Aut}(Q)$ are counted uniformly because

$$H^1(Q, V) = 0$$

in the coprime situation. Finally, invariant theory for the finite action of $\operatorname{Aut}(Q)$ recovers the orbit of the multiplicity vector of V. The right profile theorem is dual. The test targets are semidirect products

$$R_{\mathbf{r}} = U_{\mathbf{r}} \rtimes Q,$$

where $U_{\mathbf{r}}$ is a bounded semisimple $\mathbb{F}_pQ$-module. Epimorphism counts from G onto $R_{\mathbf{r}}$ count surjective module maps

$$O_p(G) \twoheadrightarrow U_{\mathbf{r}}$$

up to twisting by automorphisms of Q. These surjection moments again recover the $\mathrm{Aut}(Q)$*-orbit of the semisimple multiplicity vector. Theorem A has a reconstruction form:*

Theorem 1.2 (Reconstruction)**.** *The finite left and right hom-count fingerprints constructed in Theorem A reconstruct the semisimple $\mathbb{F}_pQ$-module $O_p(G)$ up to twisting by* $\mathrm{Aut}(Q)$*. Hence they reconstruct G up to isomorphism. We also show that the phenomenon is sharp. Let $\mathcal{A}_p$ be the class of all finite abelian p-groups, and let $\mathcal{A}_{p,\le R}$ be the subclass of exponent at most p^R.*

Theorem 1.3 (Abelian threshold)**.** *For every $R \geq 1$,*

$$\Lambda_L(\mathcal{A}_{p,\le R}) < \infty.$$

Indeed, the tests

$$C_p, C_{p^2}, \ldots, C_{p^R}$$

classify $\mathcal{A}_{p,\le R}$. In contrast,

$$\Lambda_L(\mathcal{A}_p) = \infty.$$

More strongly, no finite family of finitely generated groups distinguishes all finite abelian p-groups by homomorphism counts. Thus bounded exponent is a genuine finite-testability threshold for abelian p-groups. We also prove a stable homocyclic extension. Let

$$\mathcal{H}_{p,R,Q}$$

be the class of groups

$$G = A \rtimes Q$$

with

$$A \cong (\mathbb{Z}/p^R\mathbb{Z})^n$$

and $p \nmid |Q|$.

Theorem 1.4 (Homocyclic coprime affine groups)**.** *For fixed p, R, Q,*

$$\Lambda_L(\mathcal{H}_{p,R,Q}) < \infty \qquad \text{and} \qquad \Lambda_R(\mathcal{H}_{p,R,Q}) < \infty.$$

The proof reduces to Theorem A using the fact that coprime representations over $\mathbb{Z}/p^R\mathbb{Z}$ are determined up to conjugacy by their reductions modulo p. Finally, we prove two positive cases of the two-object hom-count test property. Let G, H be finite groups. The pair (G,H) is a left test family for the pair if

$$|\mathrm{Hom}(G,G)| = |\mathrm{Hom}(G,H)|$$

and

$$|\mathrm{Hom}(H,G)| = |\mathrm{Hom}(H,H)|$$

imply $G \cong H$. It is a right test family for the pair if

$$|\mathrm{Hom}(G,G)| = |\mathrm{Hom}(H,G)|$$

and

$$|\operatorname{Hom}(G,H)| = |\operatorname{Hom}(H,H)|$$

imply $G \cong H$*.*

Theorem 1.5 (Two-object tests)**.** *Finite abelian groups satisfy both two-object test properties. Moreover, let* Q *be a finite* p'*-group such that*

$$\operatorname{End}(Q) = \operatorname{Aut}(Q) \cup \{0\}.$$

Let

$$G_x = V_x \rtimes Q, \qquad G_y = V_y \rtimes Q,$$

where V_x, V_y *are semisimple* $\mathbb{F}_p Q$*-modules with*

$$V_x^Q = V_y^Q = 0.$$

Then G_x, G_y *satisfy both the left and right two-object test properties. The proof uses the formula*

$$|\operatorname{Hom}(G_x, G_y)| = 1 + p^{\dim V_y} \sum_{\alpha \in \operatorname{Aut}(Q)} p^{\langle x, \alpha y \rangle_D},$$

where x, y *are semisimple multiplicity vectors and* $\langle -, - \rangle_D$ *is the Schur-endomorphism-weighted inner product. The orbit-averaged exponential kernel*

$$K(x,y) = \sum_{\alpha \in \operatorname{Aut}(Q)} p^{\langle x, \alpha y \rangle_D}$$

is positive definite and separates $\operatorname{Aut}(Q)$*-orbits. The paper is organized as follows. Section 2 defines finite Lovász dimension. Section 3 treats finite abelian* p*-groups and sharpness. Sections 4–6 prove the left profile theorem for coprime affine groups. Section 7 proves the right profile theorem. Section 8 treats homocyclic affine groups. Section 9 proves the two-object theorems. Section 10 records logical and algorithmic consequences.*

2 Finite Lovász dimension for classes of groups

All groups in the paper are finite unless explicitly stated otherwise.

Definition 2.1 (Left finite Lovász dimension)**.** *Let* $\mathcal{C}$ *be a class of finite groups. We say that*

$$\Lambda_L(\mathcal{C}) \leq B$$

if for all $G, H \in \mathcal{C}$*,*

$$(|\operatorname{Hom}(T,G)| = |\operatorname{Hom}(T,H)| \text{ for every finite group } T \text{ with } |T| \leq B)$$

implies

$$G \cong H.$$

If no such B *exists, we write*

$$\Lambda_L(\mathcal{C}) = \infty.$$

Equivalently, $\Lambda_L(\mathcal{C}) < \infty$ if and only if there exists a finite family of finite groups $\mathcal{T}$ such that

$$G \cong H \iff |\operatorname{Hom}(T, G)| = |\operatorname{Hom}(T, H)| \quad \forall T \in \mathcal{T}.$$

Definition 2.2 (Right finite Lovász dimension)**.** *Similarly,*

$$\Lambda_R(\mathcal{C}) \le B$$

if for all $G, H \in \mathcal{C}$,

$$(|\operatorname{Hom}(G, R)| = |\operatorname{Hom}(H, R)| \text{ for every finite group } R \text{ with } |R| \le B)$$

implies

$$G \cong H.$$

Definition 2.3 (Finite hom-count fingerprint)**.** *A finite family $\mathcal{T}$ defines a left hom-count fingerprint*

$$\Phi^L_{\mathcal{T}}(G) = (|\operatorname{Hom}(T, G)|)_{T \in \mathcal{T}}.$$

A finite family $\mathcal{R}$ defines a right hom-count fingerprint

$$\Phi^R_{\mathcal{R}}(G) = (|\operatorname{Hom}(G, R)|)_{R \in \mathcal{R}}.$$

If $\Phi^L_{\mathcal{T}}$ is injective on isomorphism classes in $\mathcal{C}$, then $\mathcal{T}$ is a left test family for $\mathcal{C}$. Similarly for right test families.

3 Abelian p-groups: finite depth and sharpness

We begin with the simplest case. It motivates the later affine results. Let $\mathcal{A}_{p,\le R}$ be the class of finite abelian p-groups of exponent at most p^R. Every $A \in \mathcal{A}_{p,\le R}$ has a unique decomposition

$$A \cong \bigoplus_{j=1}^{R} C_{p^j}^{m_j}.$$

For $1 \le e \le R$,

$$|\operatorname{Hom}(C_{p^e}, A)| = |A[p^e]|.$$

Now

$$|A[p^e]| = p^{a_e},$$

where

$$a_e = \sum_{j=1}^{R} m_j \min(e, j).$$

Thus

$$a_e - a_{e-1} = \sum_{j \ge e} m_j.$$

It follows that

$$m_e = (a_e - a_{e-1}) - (a_{e+1} - a_e)$$

for $e < R$, and

$$m_R = a_R - a_{R-1}.$$

Proposition 3.1 (Bounded-exponent abelian p-groups)**.** *For every $R \ge 1$,*

$$\Lambda_L(\mathcal{A}_{p,\le R}) \le p^R.$$

Indeed, the finite family

$$C_p, C_{p^2}, \dots, C_{p^R}$$

classifies $\mathcal{A}_{p,\le R}$ by homomorphism counts.

Proof. The counts

$$|\operatorname{Hom}(C_{p^e}, A)|, \qquad e = 1, \dots, R,$$

recover the numbers $a_e = \log_p |A[p^e]|$, and the difference formula above recovers all multiplicities m_j. Hence they recover A.

□

The bounded-exponent condition is necessary for finite left testability in this broad abelian class.

Theorem 3.2 (No finite test family for all abelian p-groups)**.** *Let*

$$\Gamma_1, \dots, \Gamma_m$$

be any finite family of finitely generated groups. Then there exist finite abelian p-groups A, B such that: $A \not\cong B$; $|A| = |B|$; for every (j),

$$|\operatorname{Hom}(\Gamma_j, A)| = |\operatorname{Hom}(\Gamma_j, B)|.$$

In particular,

$$\Lambda_L(\mathcal{A}_p) = \infty,$$

where $\mathcal{A}_p$ is the class of all finite abelian p-groups.

Proof. Since A, B will be abelian, only the abelianizations of the Γ_j matter. Write

$$\Gamma_j^{\mathrm{ab}} \cong \mathbb{Z}^{r_j} \oplus F_j.$$

Choose D so that the p-primary component of every F_j has exponent at most p^D. Define

$$A = C_{p^{D+1}} \oplus C_{p^{D+1}},$$

$$B = C_{p^{D+2}} \oplus C_{p^D}.$$

Then

$$|A| = |B| = p^{2D+2},$$

but

$$A \ncong B.$$

For every $e \le D$,

$$|A[p^e]| = p^{2e} = |B[p^e]|.$$

Therefore

$$|\operatorname{Hom}(F_j, A)| = |\operatorname{Hom}(F_j, B)|$$

for every (j). The free part contributes

$$|A|^{r_j} = |B|^{r_j}.$$

Thus

$$|\operatorname{Hom}(\Gamma_j, A)| = |\operatorname{Hom}(\Gamma_j, B)|$$

for all (j).

□

This shows that finite Lovász dimension detects a genuine structural boundary: finite abelian p-groups of bounded exponent are finitely testable, but arbitrary finite abelian p-groups are not.

4 Coprime affine groups and semisimple modules

Fix a prime p and a finite p'-group Q. Let $\mathcal{C}_{p,Q}$ be the class of groups G satisfying

$$V_G := O_p(G) \text{ is elementary abelian,}$$

and

$$G/V_G \cong Q.$$

Since V_G is characteristic, it is intrinsic. By Schur–Zassenhaus,

$$G \cong V_G \rtimes Q.$$

Since $p \nmid |Q|$, the algebra

$$\Lambda = \mathbb{F}_p Q$$

is semisimple. Let

$$1 = e_1 + \cdots + e_s$$

be the decomposition of (1) into primitive central idempotents of Λ. Let S_i be the simple Λ-module corresponding to e_i, and set

$$d_i = \dim_{\mathbb{F}_p} S_i.$$

Every finite-dimensional Λ-module decomposes uniquely as

$$V \cong \bigoplus_{i=1}^{s} S_i^{m_i}.$$

The idempotent e_i projects onto the S_i-isotypic component. Hence

$$|\ker(e_i : V \to V)| = p^{n-m_i d_i},$$

where

$$n = \dim_{\mathbb{F}_p} V.$$

The group $\mathrm{Aut}(Q)$ acts on Λ, permuting the primitive central idempotents e_i, and therefore permuting the simple isotypic labels.

Lemma 4.1 (Isomorphism criterion)**.** *Let*

$$G = V \rtimes_\rho Q, \qquad H = W \rtimes_\sigma Q$$

belong to $\mathcal{C}_{p,Q}$. Then

$$G \cong H$$

if and only if the semisimple $\mathbb{F}_pQ$-modules V and W are isomorphic after twisting by an automorphism of Q. Equivalently, if

$$V \cong \bigoplus_i S_i^{m_i}, \qquad W \cong \bigoplus_i S_i^{n_i},$$

then $G \cong H$ if and only if the multiplicity vectors (m_i) and (n_i) lie in the same $\mathrm{Aut}(Q)$*-orbit.*

Proof. An isomorphism $G \to H$ sends $O_p(G)$ onto $O_p(H)$, because O_p is characteristic. It therefore induces an isomorphism

$$V \to W$$

and an automorphism

$$G/V \to H/W,$$

which, after identifying both quotients with Q, is an element of $\mathrm{Aut}(Q)$. This gives the twisted module isomorphism. Conversely, if $S : V \to W$ is a linear isomorphism and $\alpha \in \mathrm{Aut}(Q)$ satisfies

$$S\rho(q)S^{-1} = \sigma(\alpha(q)),$$

then

$$(v, q) \mapsto (Sv, \alpha(q))$$

defines an isomorphism

$$V \rtimes_\rho Q \cong W \rtimes_\sigma Q.$$

□

We shall reconstruct the multiplicity vector up to this automorphism action using homomorphism counts.

5 The left test family

Let $N \lhd Q$, and put

$$R_N = Q/N.$$

Let

$$\pi_N : \mathbb{F}_p Q \to \mathbb{F}_p R_N$$

be the induced algebra map. Write

$$\bar{e}_i^N = \pi_N(e_i).$$

For a tuple

$$\mathbf{i} = (i_1, \ldots, i_d),$$

define the $\mathbb{F}_p R_N$-module

$$U_{N,\mathbf{i}} \bigoplus_{\ell=1}^{d} \mathbb{F}_p R_N / \mathbb{F}_p R_N \bar{e}_{i_\ell}^N.$$

Define the finite group

$$T_{N,\mathbf{i}} U_{N,\mathbf{i}} \rtimes R_N.$$

For the empty tuple, put

$$T_{N,\varnothing} = R_N.$$

We will use only tuples with

$$d \le |\operatorname{Aut}(Q)|.$$

Thus

$$|T_{N,\mathbf{i}}| \le |Q| p^{|Q||\operatorname{Aut}(Q)|}.$$

Lemma 5.1 (Homomorphism count formula)**.** *Let $G \in \mathcal{C}_{p,Q}$, and let $V = O_p(G)$. Then*

$$|\operatorname{Hom}(T_{N,\mathbf{i}}, G)| \sum_{\theta : R_N \to G} \prod_{\ell=1}^{d} \left| \ker\left(\bar{e}_{i_\ell}^N : V_\theta \to V_\theta\right) \right|,$$

where V_θ denotes V as an $\mathbb{F}_p R_N$-module via the conjugation action induced by θ.

Proof. Every element of order dividing p in G lies in V, because its image in $G/V \cong Q$ has order dividing p, and Q is a p'-group. Therefore the elementary abelian normal subgroup $U_{N,\mathbf{i}}$ of $T_{N,\mathbf{i}}$ must map into V. Fix an homomorphism

$$\theta : R_N \to G.$$

The extensions of θ to homomorphisms

$$T_{N,\mathbf{i}} \to G$$

are precisely the $\mathbb{F}_p R_N$-module homomorphisms

$$U_{N,\mathbf{i}} \to V_\theta.$$

For a summand

$$\mathbb{F}_p R_N / \mathbb{F}_p R_N \bar{e}_i^N,$$

a module homomorphism to V_θ is determined by the image of (1), and this image may be any vector annihilated by $\bar{e}_i^N$. Thus the number of choices is

$$|\ker(\bar{e}_i^N : V_\theta \to V_\theta)|.$$

Taking the product over the direct summands and summing over θ proves the formula.

□

6 Möbius inversion and orbit recovery

Let $\mathcal{N}(Q)$ be the lattice of normal subgroups of Q, ordered by inclusion. Let

$$\mu_Q$$

be its Möbius function. For a tuple $\mathbf{i}$, define

$$E_{\mathbf{i}}(G) = \sum_{N \lhd Q} \mu_Q(1, N) |\operatorname{Hom}(T_{N,\mathbf{i}}, G)|.$$

Lemma 6.1 (Isolation of embeddings)**.** *For $G \in \mathcal{C}_{p,Q}$,*

$$E_{\mathbf{i}}(G) \sum_{\theta : Q \hookrightarrow G} \prod_{\ell=1}^{d} |\ker(e_{i_\ell} : V_\theta \to V_\theta)|.$$

Proof. An homomorphism $R_N = Q/N \to G$ is the same as an homomorphism $Q \to G$ whose kernel contains N. Therefore Lemma 5.1 expresses the count for N as a sum over homomorphisms $Q \to G$ with kernel containing N. Möbius inversion over the normal subgroup lattice isolates the terms with kernel exactly (1), i.e. the embeddings $Q \hookrightarrow G$.

□

Now choose a splitting

$$G = V \rtimes_\rho Q.$$

Every embedding

$$\theta : Q \hookrightarrow G$$

projects isomorphically onto $G/V \cong Q$, and therefore induces an automorphism

$$\alpha_\theta \in \mathrm{Aut}(Q).$$

For a fixed $\alpha \in \mathrm{Aut}(Q)$, the embeddings inducing α are parametrized by (1)-cocycles

$$Z^1(Q, V_{\rho\circ\alpha}).$$

Since $p \nmid |Q|$,

$$H^1(Q, V_{\rho\circ\alpha}) = 0.$$

Lemma 6.2 (Uniform embedding count)**.** *For every $\alpha \in \mathrm{Aut}(Q)$, the number of embeddings $Q \hookrightarrow G$ inducing α is*

$$c_G = \frac{|V|}{|V^Q|}.$$

In particular, it is independent of α.

Proof. Every (1)-cocycle is a coboundary. Explicitly, a coboundary has the form

$$z(q) = v - \rho(\alpha(q))v.$$

Two vectors (v,v') define the same coboundary if and only if

$$v - v' \in V^Q.$$

Thus

$$|Z^1(Q, V_{\rho\circ\alpha})||B^1(Q, V_{\rho\circ\alpha})|\frac{|V|}{|V^Q|}.$$

□

Define

$$K_i(G) = |\ker(e_i : V \to V)|.$$

Proposition 6.3 (Orbital moment formula)**.** *For every tuple*

$$\mathbf{i} = (i_1, \ldots, i_d),$$

$$E_{\mathbf{i}}(G)c_G \sum_{\alpha\in\mathrm{Aut}(Q)} \prod_{\ell=1}^{d} K_{\alpha(i_\ell)}(G).$$

For the empty tuple,

$$E_{\varnothing}(G) = c_G|\operatorname{Aut}(Q)|.$$

Thus the ratios

$$\frac{E_{\mathbf{i}}(G)}{E_{\varnothing}(G)}$$

recover the orbital averages of monomials in the vector

$$K(G) = (K_1(G), \dots, K_s(G)).$$

Proof. If an embedding induces α, then the induced Q-action on V is $\rho \circ \alpha$. Hence

$$|\ker(e_i : V_\theta \to V_\theta)| = K_{\alpha(i)}(G).$$

For each α there are c_G such embeddings, by Lemma 6.2. Summing over α gives the formula. □

We now record the invariant-theoretic separation step in a self-contained form.

Lemma 6.4 (Orbital monomials of bounded degree separate finite permutation orbits)**.** *Let A be a finite group acting by permutations on $1, \dots, s$. Let $x, y \in \mathbb{Q}^s$. Suppose that, for every monomial (M) of total degree at most $|A|$,*

$$\sum_{\alpha \in A} M(\alpha x) = \sum_{\alpha \in A} M(\alpha y).$$

Then x and y lie in the same A-orbit.

Proof. Let

$$\Omega_x = \{\alpha x : \alpha \in A\}, \qquad \Omega_y = \{\alpha y : \alpha \in A\},$$

counted with multiplicity. Each multiset has size $|A|$. The equality of all monomial orbit sums of degree at most $|A|$ implies that for every linear form ℓ, the power sums

$$\sum_{z \in \Omega_x} \ell(z)^r$$

and

$$\sum_{z \in \Omega_y} \ell(z)^r$$

are equal for all

$$0 \le r \le |A|.$$

For a finite multiset of size $|A|$ in a field of characteristic zero, the first $|A|$ power sums determine the multiset of values. Hence

$$\{\ell(z) : z \in \Omega_x\} = \{\ell(z) : z \in \Omega_y\}$$

for every linear form ℓ. Choose ℓ injective on the finite set $\Omega_x \cup \Omega_y$. Then equality of the multisets of ℓ-values implies

$$\Omega_x = \Omega_y.$$

Thus x and y lie in the same A-orbit.

$\square$

Theorem 6.5 (Left finite Lovász dimension for $\mathcal{C}_{p,Q}$). *For every prime p and finite p'-group Q,*

$$\Lambda_L(\mathcal{C}_{p,Q}) \leq |Q| p^{|Q||\operatorname{Aut}(Q)|}.$$

More precisely, the finite family consisting of C_p and the groups

$$T_{N,\mathbf{i}}, \qquad N \lhd Q, \quad |\mathbf{i}| \leq |\operatorname{Aut}(Q)|,$$

classifies $\mathcal{C}_{p,Q}$ by left homomorphism counts.

Proof. The count

$$|\operatorname{Hom}(C_p, G)|$$

is the number of elements $g \in G$ with $g^p = 1$. Since $G/O_p(G) \cong Q$ is a p'-group and $O_p(G)$ is elementary abelian,

$$|\operatorname{Hom}(C_p, G)| = |O_p(G)| = p^n.$$

The counts from the groups $T_{N,\mathbf{i}}$, via Möbius inversion and Proposition 6.3, recover all orbital monomial sums of degree at most $|\operatorname{Aut}(Q)|$ in the vector

$$K(G) = (K_i(G)).$$

By Lemma 6.4, they recover the $\operatorname{Aut}(Q)$-orbit of (K(G)). Since

$$K_i(G) = p^{n - m_i d_i},$$

and (n) is known, the orbit of (K(G)) recovers the orbit of the multiplicity vector

$$(m_1, \ldots, m_s).$$

By Lemma 4.1, this orbit determines G up to isomorphism.

$\square$

This proves the left half of Theorem A and the reconstruction statement for the left profile.

7 The right test family

We now prove the right profile analogue. Let

$$V_m = \bigoplus_{i=1}^{s} S_i^{m_i}$$

be a semisimple $\mathbb{F}_p Q$-module, and let

$$G_m = V_m \rtimes Q.$$

For a vector

$$\mathbf{r} = (r_1, \ldots, r_s) \in \mathbb{N}^s,$$

define

$$U_{\mathbf{r}} = \bigoplus_{i=1}^{s} S_i^{r_i},$$

and

$$R_{\mathbf{r}} = U_{\mathbf{r}} \rtimes Q.$$

We shall use only

$$|\mathbf{r}| := r_1 + \cdots + r_s \le |\operatorname{Aut}(Q)|.$$

Also include all subgroups of the finitely many $R_{\mathbf{r}}$ in the test family. Let

$$D_i = \operatorname{End}_Q(S_i),$$

a finite division ring. Put

$$q_i = |D_i|.$$

For $m, r \ge 0$, define

$$\operatorname{Surj}_{D_i}(m, r) = \#\{\text{surjective } D_i\text{-linear maps } D_i^m \to D_i^r\}.$$

Thus

$$\operatorname{Surj}_{D_i}(m, r) = \prod_{j=0}^{r-1} (q_i^m - q_i^j),$$

with the convention that the empty product is (1).

Lemma 7.1 (Epimorphism count formula)**.** *Let*

$$G_m = V_m \rtimes Q.$$

Then

$$|\operatorname{Epi}(G_m, R_{\mathbf{r}})| c_{\mathbf{r}} \sum_{\alpha \in \operatorname{Aut}(Q)} \prod_{i=1}^{s} \operatorname{Surj}_{D_i}(m_{\alpha i}, r_i),$$

where $c_{\mathbf{r}} > 0$ depends only on Q and $\mathbf{r}$, not on (m).

Proof. An epimorphism

$$G_m \twoheadrightarrow R_{\mathbf{r}}$$

induces an epimorphism

$$Q \to Q$$

on the quotients by the normal p-subgroups. Since domain and codomain quotients are both Q, this induced map is an automorphism

$$\alpha \in \mathrm{Aut}(Q).$$

Fix α. The restriction to the p-kernel is a Q-module map

$$V_m \to U_{\mathbf{r}}^{\alpha},$$

where the target is twisted by α. The total group homomorphism is surjective if and only if this module map is surjective. The number of surjective module maps is

$$\prod_i \mathrm{Surj}_{D_i}(m_{\alpha i}, r_i).$$

For each α, the possible images of the complement are parametrized by

$$Z^1(Q, U_{\mathbf{r}}^{\alpha}).$$

Since $p \nmid |Q|$,

$$H^1(Q, U_{\mathbf{r}}^{\alpha}) = 0,$$

so the number of cocycles is

$$|Z^1(Q, U_{\mathbf{r}}^{\alpha})| \frac{|U_{\mathbf{r}}|}{|U_{\mathbf{r}}^Q|}.$$

This depends only on $\mathbf{r}$, not on (m) or α. Absorbing this factor into $c_{\mathbf{r}}$ gives the formula.

□

To recover epimorphism counts from homomorphism counts, we use the elementary triangular relation.

Lemma 7.2 (Hom-to-epi inversion)**.** *Let R be a finite group. If the values*

$$|\,\mathrm{Hom}(G, S)|$$

are known for every subgroup $S \leq R$, then

$$|\,\mathrm{Epi}(G, S)|$$

is known for every subgroup $S \leq R$.

Proof. Every homomorphism $G \to S$ has image a subgroup $U \leq S$, and it is an epimorphism onto its image. Therefore

$$|\operatorname{Hom}(G,S)| = \sum_{U \leq S} |\operatorname{Epi}(G,U)|.$$

Induction on $|S|$ recovers $|\operatorname{Epi}(G,S)|$.

□

The functions

$$\operatorname{Surj}_{D_i}(m,r) = \prod_{j=0}^{r-1} (q_i^m - q_i^j)$$

are polynomial functions of

$$X_i = q_i^{m_i}$$

of degree (r), with leading coefficient (1). Therefore the products

$$\prod_i \operatorname{Surj}_{D_i}(m_i, r_i)$$

span the same polynomial space as monomials

$$\prod_i X_i^{r_i}$$

of total degree at most $|\operatorname{Aut}(Q)|$. By Lemma 6.4, the orbital sums of these polynomials recover the $\operatorname{Aut}(Q)$-orbit of

$$(X_i) = (q_i^{m_i}).$$

Since q_i is fixed and positive, this recovers the orbit of

$$(m_i).$$

Theorem 7.3 (Right finite Lovász dimension for $\mathcal{C}_{p,Q}$)**.** *For every prime p and finite p'-group Q,*

$$\Lambda_R(\mathcal{C}_{p,Q}) < \infty.$$

More precisely, the family consisting of all subgroups of the finitely many groups

$$R_{\mathbf{r}} = U_{\mathbf{r}} \rtimes Q, \qquad |\mathbf{r}| \leq |\operatorname{Aut}(Q)|,$$

classifies $\mathcal{C}_{p,Q}$ by right homomorphism counts.

Proof. Given the right homomorphism counts into all subgroups of the $R_{\mathbf{r}}$, Lemma 7.2 recovers the epimorphism counts onto the $R_{\mathbf{r}}$. By Lemma 7.1, these epimorphism counts give the orbital sums

$$\sum_{\alpha \in \operatorname{Aut}(Q)} \prod_i \operatorname{Surj}_{D_i}(m_{\alpha i}, r_i)$$

for all $|\mathbf{r}| \leq |\operatorname{Aut}(Q)|$. These span the orbital monomial sums of degree at most $|\operatorname{Aut}(Q)|$ in the variables $q_i^{m_i}$. By Lemma 6.4, they recover the $\operatorname{Aut}(Q)$-orbit of the multiplicity vector (m_i). By Lemma 4.1, this orbit determines G.

□

Combining Theorem 6.5 and Theorem 7.3 proves Theorem A.

8 Homocyclic coprime affine groups

We now extend the finite testability theorem to homocyclic p-kernels. Fix $R \geq 1$. Let $\mathcal{H}_{p,R,Q}$ be the class of groups

$$G = A \rtimes Q$$

such that

$$A \cong (\mathbb{Z}/p^R\mathbb{Z})^n$$

for some (n), and

$$p \nmid |Q|.$$

The key point is that coprime representations over $\mathbb{Z}/p^R\mathbb{Z}$ are determined by their reductions modulo p.

Lemma 8.1 (Coprime lifting rigidity)**.** *Let*

$$\rho, \sigma : Q \to GL_n(\mathbb{Z}/p^R\mathbb{Z})$$

be representations of a finite p'-group Q. If their reductions modulo p are conjugate in $GL_n(\mathbb{F}_p)$, then ρ and σ are conjugate in

$$GL_n(\mathbb{Z}/p^R\mathbb{Z}).$$

Proof. After an initial conjugation, assume

$$\rho \equiv \sigma \pmod{p}.$$

We prove by induction that they are conjugate modulo p^r for $r = 1, \ldots, R$. Suppose they are equal modulo p^r. Modulo p^{r+1}, write

$$\sigma(q) = (I + p^r c(q))\rho(q).$$

Then

$$c : Q \to M_n(\mathbb{F}_p)$$

is a (1)-cocycle for the action

$$q \cdot X = \bar{\rho}(q) X \bar{\rho}(q)^{-1}.$$

Since $p \nmid |Q|$,

$$H^1(Q, M_n(\mathbb{F}_p)) = 0.$$

Hence

$$c(q) = X - q \cdot X$$

for some $X \in M_n(\mathbb{F}_p)$. Conjugating by

$$I + p^r X$$

removes the error modulo p^{r+1}. Induction gives the desired conjugacy modulo p^R.

□

Theorem 8.2 (Homocyclic coprime affine groups)**.** *For fixed p, R, Q with $p \nmid |Q|$,*

$$\Lambda_L(\mathcal{H}_{p,R,Q}) < \infty \qquad \textit{and} \qquad \Lambda_R(\mathcal{H}_{p,R,Q}) < \infty.$$

Proof. Let

$$G = A \rtimes Q, \qquad A \cong (\mathbb{Z}/p^R\mathbb{Z})^n.$$

The subgroup

$$A[p] = a \in A : pa = 0$$

is an elementary abelian characteristic subgroup, naturally isomorphic as an $\mathbb{F}_pQ$-module to the reduction of A modulo p. The left and right test families of Theorem 6.5 and Theorem 7.3, applied to the elementary module (A[p]), recover the semisimple $\mathbb{F}_pQ$-module (A[p]) up to $\mathrm{Aut}(Q)$-twist. By Lemma 8.1, the full $(\mathbb{Z}/p^R\mathbb{Z})Q$-module A is determined by this reduction. Hence the group $A \rtimes Q$ is determined. Only bounded-size tests depending on p, R, Q are involved.

□

9 Two-object hom-count tests

We now prove two positive cases of the two-object test property.

9.1 Finite abelian groups

Let A be a finite abelian group. For each prime p, write

$$A_p \cong \bigoplus_{e \geq 1} C_{p^e}^{m_{p,e}(A)}.$$

Define

$$u_{p,e}(A) = \sum_{j \geq e} m_{p,j}(A).$$

Thus $u_{p,e}(A)$ is the number of cyclic p-primary summands of A of order at least p^e. For finite abelian groups A, B,

$$|\operatorname{Hom}(A,B)| = \prod_p p^{\sum_{e\geq 1} u_{p,e}(A)u_{p,e}(B)}.$$

Equivalently,

$$\log|\operatorname{Hom}(A,B)| = \sum_{p,e}(\log p), u_{p,e}(A)u_{p,e}(B).$$

This is an inner product.

Theorem 9.1 (Two-object tests for finite abelian groups)**.** *Finite abelian groups satisfy both the left and right two-object test properties. That is, for finite abelian groups A, B, either of the following pairs of equalities implies $A \cong B$: Left-profile equalities:*

$$|\operatorname{Hom}(A,A)| = |\operatorname{Hom}(A,B)|,$$

$$|\operatorname{Hom}(B,A)| = |\operatorname{Hom}(B,B)|.$$

Right-profile equalities:

$$|\operatorname{Hom}(A,A)| = |\operatorname{Hom}(B,A)|,$$

$$|\operatorname{Hom}(A,B)| = |\operatorname{Hom}(B,B)|.$$

Proof. For finite abelian groups,

$$|\operatorname{Hom}(A,B)| = |\operatorname{Hom}(B,A)|.$$

Taking logarithms identifies the homomorphism count with the exponential of an inner product of the vectors

$$u(A) = (u_{p,e}(A))_{p,e}.$$

Either pair of equalities gives

$$\langle u(A), u(A)\rangle\langle u(A), u(B)\rangle\langle u(B), u(B)\rangle.$$

By equality in Cauchy–Schwarz, $u(A) = u(B)$. The vector $u(A)$ determines the multiplicities $m_{p,e}(A)$, since

$$m_{p,e}(A) = u_{p,e}(A) - u_{p,e+1}(A).$$

Hence $A \cong B$.

□

9.2 Coprime affine two-object theorem

Let Q be a finite p'-group satisfying

$$\operatorname{End}(Q) = \operatorname{Aut}(Q) \cup \{0\},$$

where (0) denotes the trivial endomorphism. Let

$$G_x = V_x \rtimes Q, \qquad G_y = V_y \rtimes Q,$$

where V_x, V_y are semisimple $\mathbb{F}_pQ$-modules with

$$V_x^Q = V_y^Q = 0.$$

Let

$$S_1, \dots, S_s$$

be the nontrivial simple $\mathbb{F}_pQ$-modules appearing in this setting. Write

$$V_x = \bigoplus_i S_i^{x_i}, \qquad V_y = \bigoplus_i S_i^{y_i}.$$

Let

$$D_i = \operatorname{End} Q(S_i), \qquad d_i = \dim \mathbb{F}_p D_i.$$

Define

$$\langle x, y\rangle_D \sum_i d_i x_i y_i.$$

The group $\operatorname{Aut}(Q)$ acts on the simple modules and preserves d_i. Define the orbit-exponential kernel

$$K(x, y) \sum_{\alpha\in\operatorname{Aut}(Q)} p^{\langle x,\alpha y\rangle_D}.$$

Lemma 9.2 (Homomorphism count formula)**.** *Under the assumptions above,*

$$|\operatorname{Hom}(G_x, G_y)| = 1 + p^{\dim V_y} K(x, y).$$

Proof. Let

$$f : G_x \to G_y$$

be a homomorphism. It induces an endomorphism

$$Q \to Q.$$

By assumption, this endomorphism is either trivial or an automorphism. If the induced endomorphism is trivial, then the image of the complement Q lies in the p-group V_y. Since Q is a p'-group, this image is trivial. The restriction $V_x \to V_y$ must factor through the coinvariants of V_x. Since $p \nmid |Q|$, coinvariants and invariants have the same dimension; since $V_x^Q = 0$, this map is zero.

Thus the trivial quotient case contributes exactly one homomorphism. Now suppose the induced endomorphism is $\alpha \in \mathrm{Aut}(Q)$. The restriction to V_x is a Q-module homomorphism

$$V_x \to V_y^{\alpha},$$

and the number of such homomorphisms is

$$p^{\langle x, \alpha y\rangle_D}.$$

The possible images of the complement are parametrized by

$$Z^1(Q, V_y^{\alpha}).$$

Since $p \nmid |Q|$,

$$H^1(Q, V_y^{\alpha}) = 0.$$

Since $V_y^Q = 0$,

$$|Z^1(Q, V_y^{\alpha})| = |V_y| = p^{\dim V_y}.$$

Summing over $\alpha \in \mathrm{Aut}(Q)$ gives the formula.

□

Lemma 9.3 (The kernel K separates orbits)**.** *Let*

$$A = \mathrm{Aut}(Q).$$

The kernel

$$K(x, y) = \sum_{\alpha \in A} p^{\langle x, \alpha y\rangle_D}$$

is positive definite after orbit averaging and separates A-orbits. In particular, if equality holds in Cauchy–Schwarz for K, then (x) and (y) lie in the same A-orbit.

Proof. Consider the real inner product space with inner product $\langle -, -\rangle_D$. Define

$$\Phi(x) \bigoplus_{r \geq 0} \frac{(\sqrt{\log p}, x)^{\otimes r}}{\sqrt{r!}}.$$

Then

$$\langle \Phi(x), \Phi(y)\rangle p^{\langle x, y\rangle_D}.$$

Now define the orbit feature map

$$Psi(x) = \sum_{\alpha \in A} \Phi(\alpha x).$$

Then

$$\langle \Psi(x), \Psi(y)\rangle |A| K(x, y).$$

If equality holds in Cauchy–Schwarz for K, then equality holds for the vectors $\Psi(x), \Psi(y)$. Hence $\Psi(x) = c\Psi(y)$ for some $c > 0$. Comparing the degree-zero component gives $c = 1$, so $\Psi(x) = \Psi(y)$. Therefore, for every vector (z),

$$\sum_{\alpha \in A} p^{\langle \alpha x, z \rangle_D} \sum_{\alpha \in A} p^{\langle \alpha y, z \rangle_D}.$$

Distinct exponential linear forms are linearly independent as functions of z. Hence the multisets

$$Ax \quad \text{and} \quad Ay$$

coincide. Thus x and y lie in the same A-orbit.

□

Theorem 9.4 (Two-object tests for endomorphism-simple coprime affine groups)**.** *Let Q be a finite p'-group satisfying*

$$\mathrm{End}(Q) = \mathrm{Aut}(Q) \cup \{0\}.$$

Let

$$G_x = V_x \rtimes Q, \qquad G_y = V_y \rtimes Q,$$

with

$$V_x^Q = V_y^Q = 0.$$

Then G_x, G_y satisfy both the left and right two-object test properties.

Proof. Write

$$h(x, y) = |\,\mathrm{Hom}(G_x, G_y)|.$$

By Lemma 9.2,

$$h(x, y) = 1 + p^{\dim V_y} K(x, y).$$

Left-profile version Assume

$$h(x, x) = h(x, y),$$

$$h(y, x) = h(y, y).$$

Then

$$p^{\dim V_x} K(x, x) = p^{\dim V_y} K(x, y),$$

and

$$p^{\dim V_x} K(x,y) = p^{\dim V_y} K(y,y).$$

Multiplying gives

$$K(x,x)K(y,y) = K(x,y)^2.$$

Thus equality holds in Cauchy–Schwarz for K. By Lemma 9.3, (x) and (y) are in the same Aut(Q)-orbit. Hence

$$G_x \cong G_y.$$

Right-profile version Assume

$$h(x,x) = h(y,x),$$

$$h(x,y) = h(y,y).$$

Then

$$K(x,x) = K(x,y),$$

and

$$K(x,y) = K(y,y).$$

Again equality holds in Cauchy–Schwarz, and Lemma 9.3 gives that x, y lie in the same Aut(Q)-orbit. Hence

$$G_x \cong G_y.$$

□

This proves a nontrivial infinite class of positive cases for the two-object hom-count test property.

10 Equation-count profiles and algorithmic remarks

For every finite test group T, the number

$$|\operatorname{Hom}(T,G)|$$

is the number of solutions in G of a fixed finite system of group equations. Indeed, introduce one variable x_t for each $t \in T$, and impose

$$x_1 = 1,$$

$$x_a x_b = x_{ab} \qquad (a, b \in T).$$

The solutions are exactly the homomorphisms $T \to G$. Therefore Theorem 6.5 has the following purely equational form.

Corollary 10.1 (Bounded equation-count profile)**.** *For fixed p, Q, the groups in $\mathcal{C}_{p,Q}$ are classified by finitely many ordinary group-equation counts of bounded size. Equivalently, if $G, H \in \mathcal{C}_{p,Q}$ are non-isomorphic, then there exists a group-equation system of size bounded in terms of p, Q whose number of solutions differs in G and H. Thus every non-isomorphic pair in $\mathcal{C}_{p,Q}$ has a bounded hom-count separator.*

Corollary 10.2 (Logarithmic fingerprints)**.** *For fixed p, Q, the finite left and right fingerprints of groups in $\mathcal{C}_{p,Q}$ have total bit length*

$$O_{p,Q}(\log |G|).$$

Proof. The number of coordinates is bounded in terms of p, Q. Each coordinate is at most

$$|G|^c$$

for a constant (c) depending only on the relevant bounded test group. Hence each coordinate has $O_{p,Q}(\log |G|)$ bits.

□

These fingerprints are not claimed to improve the best known algorithms for isomorphism testing in this regime. Rather, they provide a fixed finite list of numerical homomorphism-count invariants that completely reconstruct the semisimple affine data.

11 Further directions

11.1 Arbitrary bounded-exponent abelian kernels

The homocyclic theorem suggests a broader statement for groups

$$A \rtimes Q$$

where A is abelian of exponent at most p^R. Proving this in full requires a careful treatment of finite modules over

$$(\mathbb{Z}/p^R\mathbb{Z})Q.$$

The expected answer is that finite Lovász dimension remains finite for fixed p, R, Q, with test groups built from block idempotents and p^e-torsion functors.

11.2 Nonabelian semisimple socles

A second direction is to replace the abelian kernel $O_p(G)$ by a centerless semisimple socle

$$S_1^{n_1} \times \cdots \times S_t^{n_t}$$

with quotient Q acting on the simple factors. The correct analogue should involve decorated table-of-marks data for Q-sets labeled by outer automorphism data of the simple factors. This would give a nonabelian version of the finite Lovász profile theorem.

11.3 Minimal test families

The constructions in this paper are explicit but not optimized. Natural questions include: determining the smallest possible B with $\Lambda_L(\mathcal{C}_{p,Q}) \leq B$; minimizing the number of test groups; understanding when a single test group can separate the whole class.

11.4 Full two-object test problem

Theorem 9.4 proves the two-object test property for affine groups with endomorphism-simple quotient and no fixed vectors. The general coprime affine case reduces to explicit formulas involving

$$\mathrm{End}(Q)$$

and twisted Hom-spaces of semisimple modules. It remains to determine for which finite groups Q the two-object test property holds on the corresponding affine class.

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