Random Saturation and Optimal Off-Origin Slack in Real Homogeneous Boolean Threshold Boxes

Luca Blanchi

June 2026

Abstract

We study bounded-height homogeneous real polynomial lifts of Boolean sign functions. A homogeneous form of degree D in variables $X_0, \ldots, X_n$, with all homogeneous coefficients bounded in absolute value by 1, induces on the affine Boolean chart

$$(X_0, X_1, \ldots, X_n) = (1, x_1, \ldots, x_n), \qquad x_i \in 0, 1,$$

a multilinear polynomial

$$p(x) = \sum_{S \subseteq [n]} a_S x_S$$

whose coefficients satisfy the exact box constraints

$$|a_S| \leq \binom{D}{|S|}.$$

Conversely, every coefficient vector satisfying these inequalities arises from a homogeneous degree-D form of coefficient height at most 1. For a Boolean sign function $G : 0, 1^n \to \pm 1$, define the real box margin

$$\gamma_D(G) = \max_{|a_S| \leq \binom{D}{|S|}} \min_{T \subseteq [n]} G_T \sum_{S \subseteq T} a_S.$$

Since the empty point sees only $a_\varnothing$, one always has

$$\gamma_D(G) \leq 1.$$

Let H_2 denote the binary entropy function, and let $\rho_* \in (1/2, 1)$ be the unique solution of

$$H_2(\rho_*) = \rho_*.$$

Numerically,

$$\rho_* = 0.7729078047\ldots, \qquad \frac{1}{2\rho_*} = 0.6469076867\ldots.$$

We prove that if D_n is a linear degree sequence satisfying

$$0 < \liminf_{n \to \infty} \frac{D_n}{n} \leq \limsup_{n \to \infty} \frac{D_n}{n} < \infty$$

and

$$\liminf_{n \to \infty} \frac{D_n}{n} > \frac{1}{2\rho_*},$$

then for uniformly random $G_n : 0, 1^n \to \pm 1$,

$$\Pr(\gamma_{D_n}(G_n) = 1) \to 1.$$

The main theorem is sharper. After fixing the empty coefficient to saturate the origin, the optimal margin on every non-origin Boolean point is asymptotically maximal:

$$\eta_{D_n}(G_n) = D_n - 1 - o(1) \qquad \text{with high probability.}$$

Thus, in this bounded-height homogeneous real model, the origin is the unique bottleneck. The proof combines exact minimax duality for boxes, a sharp weighted upward-zeta localization estimate, a complement transform converting high-zeta data into low-degree Boolean polynomials, and a projection-separation lemma showing that a random signed simplex is far in ℓ^1 from every sufficiently low-dimensional polynomial subspace. All results in this paper are over real coefficients. They do not imply integer, ternary, or rounded coefficient representations.

1 Introduction

A classical question in Boolean function analysis asks for the degree needed to sign-represent a Boolean function by a real polynomial. In the unbounded-coefficient setting, random Boolean functions have threshold degree near $n/2$. This paper studies a different model: real polynomial representations arising from homogeneous forms of bounded coefficient height. Let

$$F \in \mathbb{R}[X_0, \dots, X_n]_D$$

be homogeneous of degree D. Restrict F to the affine Boolean chart

$$(X_0, X_1, \dots, X_n) = (1, x_1, \dots, x_n), \qquad x_i \in 0, 1,$$

and reduce by the Boolean relations

$$x_i^2 = x_i.$$

If every homogeneous coefficient of F has absolute value at most 1, the resulting multilinear polynomial

$$p(x) = \sum_{S \subseteq [n]} a_S x_S$$

satisfies

$$|a_S| \le \binom{D}{|S|}.$$

Conversely, every multilinear coefficient vector satisfying these inequalities is realized by some homogeneous degree-D form with coefficient height at most 1. Thus the coefficient body induced by height-one homogeneous degree-D forms is exactly

$$\mathcal{B}_{n,D} \left\{ a = (a_S)_{S \subseteq [n]} : |a_S| \le \binom{D}{|S|} \right\},$$

with the convention

$$\binom{D}{s} = 0 \qquad (s > D).$$

For $G : 0,1^n \to \pm 1$, define

$$\gamma_D(G) = \max_{a \in \mathcal{B}_{n,D}} \min_{T \subseteq [n]} G_T \sum_{S \subseteq T} a_S.$$

At the empty point $T = \varnothing$, the polynomial value is simply $a_\varnothing$. Since

$$|a_\varnothing| \le 1,$$

one always has

$$\gamma_D(G) \le 1.$$

We prove that this upper bound is attained with high probability for random G once

$$D > (0.646907\ldots)n.$$

More precisely, in the linear degree regime

$$D_n = \Theta(n),$$

if

$$\liminf_{n\to\infty} \frac{D_n}{n} > \frac{1}{2\rho_*},$$

then

$$\gamma_{D_n}(G_n) = 1$$

with high probability. The theorem is stronger than saturation. After fixing $a_\varnothing$ to saturate the origin, the optimal margin on every nonempty Boolean point is

$$D_n - 1 - o(1),$$

which is asymptotically the largest possible margin forced by singleton constraints. This is a bounded-coefficient random representation theorem for a concrete homogeneous coefficient body. It is not an unbounded-coefficient threshold-degree theorem.

2 Homogeneous coefficient boxes

We identify a Boolean vector $x \in 0,1^n$ with its support

$$T = i : x_i = 1 \subseteq [n].$$

For $S \subseteq [n]$, write

$$x_S = \prod_{i \in S} x_i.$$

Let

$$F \in \mathbb{R}[X_0, \ldots, X_n]_D$$

be homogeneous of degree D. A monomial

$$X_0^{\alpha_0} X_1^{\alpha_1} \cdots X_n^{\alpha_n}, \qquad \sum_{i=0}^{n} \alpha_i = D,$$

reduces on the affine Boolean chart to x_S precisely when

$$\alpha_i \geq 1 \quad (i \in S),$$

and

$$\alpha_j = 0 \quad (j \notin S,\ j \geq 1).$$

Thus the number of homogeneous degree-D monomials reducing to x_S is the number of solutions

$$\alpha_0 + \sum_{i \in S} \alpha_i = D, \qquad \alpha_0 \geq 0, \qquad \alpha_i \geq 1 \quad (i \in S).$$

Equivalently, after writing

$$\beta_i = \alpha_i - 1 \quad (i \in S),$$

we count nonnegative solutions of

$$\alpha_0 + \sum_{i \in S} \beta_i = D - |S|.$$

The number is

$$\binom{D}{|S|}.$$

Therefore, if every homogeneous coefficient of F lies in ([-1,1]), the induced Boolean coefficient a_S satisfies

$$|a_S| \leq \binom{D}{|S|}.$$

Conversely, suppose

$$|a_S| \leq \binom{D}{|S|}$$

for every $S \subseteq [n]$. For each S, distribute the real number a_S evenly among the $\binom{D}{|S|}$ homogeneous monomials reducing to x_S. Each assigned homogeneous coefficient has absolute value at most 1, and the resulting homogeneous degree-D form restricts to the given multilinear polynomial on the Boolean cube. Hence the exact real coefficient box induced by homogeneous degree-D, height-one forms is

$$\mathcal{B}_{n,D} \left\{ a = (a_S)_{S \subseteq [n]} : |a_S| \leq \binom{D}{|S|} \right\}.$$

3 Margins and off-origin slack

Let

$$G : 0, 1^n \to \pm 1,$$

and write

$$G_T = G(1_T).$$

Define the full box margin

$$\gamma_D(G) = \max_{|a_S| \le \binom{D}{|S|}} \min_{T \subseteq [n]} G_T \sum_{S \subseteq T} a_S.$$

As noted above,

$$\gamma_D(G) \le 1.$$

Because the coefficient box is centrally symmetric,

$$\gamma_D(G) = \gamma_D(-G).$$

Thus we may normalize

$$G_\varnothing = 1$$

when proving lower bounds. For unnormalized G, define the off-origin margin by fixing the empty coefficient to saturate the origin:

$$\eta_D(G) \max_{|a_S| \le \binom{D}{|S|}} \min_{a_\varnothing = G_\varnothing} \min_{\varnothing \ne T \subseteq [n]} G_T \sum_{S \subseteq T} a_S.$$

After multiplying G by $G_\varnothing$, this is equivalent to the normalized setting

$$G_\varnothing = 1, \qquad a_\varnothing = 1.$$

If

$$\eta_D(G) \ge 1,$$

then

$$\gamma_D(G) = 1.$$

Indeed, the empty point has margin exactly 1, and every nonempty point has margin at least 1. The main theorem proves much more:

$$\eta_{D_n}(G_n) = D_n - 1 - o(1)$$

with high probability for random G_n, in the linear degree regime above the threshold

$$\frac{1}{2\rho_*}.$$

4 Exact minimax duality

We first record a general box-duality statement. Let

$$B = (B_S)S \subseteq [n]$$

be a nonnegative weight system, and define

$$\gamma_B(G) = \max_{|a_S| \le B_S} \min_{T \subseteq [n]} G_T \sum_{S \subseteq T} a_S.$$

Let

$$\Delta(2^{[n]})$$

denote the probability simplex on all subsets of ([n]). Theorem 4.1. Box duality For every $G : 0, 1^n \to \pm 1$,

$$\gamma_B(G) \min_{\lambda \in \Delta(2^{[n]})} \sum_{S \subseteq [n]} B_S \left| \sum_{T \supseteq S} \lambda_T G_T \right|.$$

Proof For fixed $a = (a_S)$, put

$$L_T(a) = G_T \sum_{S \subseteq T} a_S.$$

Then

$$\min_T L_T(a) \min_{\lambda \in \Delta(2^{[n]})} \sum_T \lambda_T L_T(a).$$

Hence

$$\gamma_B(G) \max_{|a_S| \le B_S} \min_{\lambda \in \Delta(2^{[n]})} \sum_T \lambda_T G_T \sum_{S \subseteq T} a_S.$$

The coefficient box and the simplex are compact convex sets, and the objective is bilinear. By finite-dimensional minimax duality,

$$\gamma_B(G) \min_{\lambda \in \Delta(2^{[n]})} \max_{|a_S| \le B_S} \sum_S a_S \sum_{T \supseteq S} \lambda_T G_T.$$

For fixed λ, the inner maximum is the support function of the box:

$$\max_{|a_S| \le B_S} \sum_S a_S c_S \sum_S B_S |c_S|.$$

Taking

$$c_S = \sum_{T \supseteq S} \lambda_T G_T$$

gives the result. □

5 Duality for off-origin slack

Assume throughout this section that

$$G_\varnothing = 1.$$

For a probability measure ν on nonempty subsets, define

$$A_G(\nu) = \sum_{T \neq \varnothing} \nu_T G_T$$

and

$$B_{D,G}(\nu) \sum_{S \neq \varnothing \ |S| \leq D} \binom{D}{|S|} \left| \sum_{\substack{T \supseteq S \\ T \neq \varnothing}} \nu_T G_T \right| .$$

Proposition 5.1. Dual form of off-origin slack For $G_\varnothing = 1$,

$$\eta_D(G) \min_{\nu \in \Delta(2^{[n]} \setminus \varnothing)} \left[A_G(\nu) + B_{D,G}(\nu) \right] .$$

Proof In the definition of η_D, the coefficient $a_\varnothing$ is fixed to 1, and the variables are a_S for $S \neq \varnothing$. For a probability measure ν on nonempty T, the averaged objective is

$$\sum_{T \neq \varnothing} \nu_T G_T \left(1 + \sum_{S \subseteq T \S \neq \varnothing} a_S \right) .$$

This equals

$$A_G(\nu) + \sum_{S \neq \varnothing} a_S \sum_{\substack{T \supseteq S \\ T \neq \varnothing}} \nu_T G_T .$$

Applying minimax over the coefficient box for the variables a_S, $S \neq \varnothing$, the maximum over

$$|a_S| \leq \binom{D}{|S|}$$

is precisely

$$B_{D,G}(\nu).$$

Thus

$$\eta_D(G) \min_\nu \left[A_G(\nu) + B_{D,G}(\nu) \right] .$$

☐ Since always

$$A_G(\nu) \geq -1,$$

a uniform lower bound on $B_{D,G}(\nu)$ gives an off-origin slack lower bound.

6 Sharp weighted upward-zeta localization

Let ν be a probability measure on nonempty subsets and define

$$\theta_T = \nu_T G_T, \qquad T \neq \varnothing.$$

Then

$$|\theta|_1 = 1.$$

Define the upward-zeta transform of θ by

$$h_S = \sum_{\substack{T \supseteq S \\ T \neq \varnothing}} \theta_T, \qquad S \neq \varnothing.$$

Then

$$B_{D,G}(\nu) \sum_{S \neq \varnothing \ |S| \leq D} \binom{D}{|S|} |h_S|.$$

For $1 \leq r \leq D$, let $h^{\leq r}$ be the restriction of h to levels

$$1 \leq |S| \leq r.$$

Let $\theta^{\leq r}$ be the Möbius inverse of this truncated zeta transform:

$$\theta_T^{\leq r} \sum_{\substack{S \supseteq T \\ 1 \leq |S| \leq r}} (-1)^{|S|-|T|} h_S, \qquad T \neq \varnothing.$$

Define

$$\mathfrak{m}^*(D, r) \max_{1 \leq s \leq r} \frac{2^s - 1}{\binom{D}{s}}.$$

Lemma 6.1. Sharp truncated inverse bound For every ν,

$$|\theta^{\leq r}|_1 \leq \mathfrak{m}^*(D, r) B_{D,G}(\nu).$$

Proof By the triangle inequality,

$$|\theta_T^{\leq r}| \leq \sum_{\substack{S \supseteq T \\ 1 \leq |S| \leq r}} |h_S|.$$

Summing over all nonempty T,

$$|\theta^{\leq r}|_1 \leq \sum_{1 \leq |S| \leq r} |T \neq \varnothing : T \subseteq S|, |h_S|.$$

For fixed $S \neq \varnothing$,

$$|T \neq \varnothing : T \subseteq S| = 2^{|S|} - 1.$$

Therefore

$$|\theta^{\leq r}|_1 \leq \sum_{1\leq |S|\leq r} (2^{|S|}-1)|h_S|.$$

Hence

$$|\theta^{\leq r}|_1 \leq \left(\max_{1\leq s\leq r} \frac{2^s-1}{\binom{D}{s}}\right) \sum_{1\leq |S|\leq r} \binom{D}{|S|} |h_S| \leq \mathfrak{m}^*(D,r) B_{D,G}(\nu).$$

□ The upward-zeta transform of $\theta^{\leq r}$ agrees with h on all nonempty levels $1 \leq |S| \leq r$ and vanishes on nonempty levels $|S| > r$. Hence, if

$$\theta^{>r} = \theta - \theta^{\leq r},$$

then the upward-zeta transform of $\theta^{>r}$ vanishes on all nonempty levels

$$1 \leq |S| \leq r.$$

7 Complement transform and low-degree polynomials

Let

$$\theta^{>r} = \theta - \theta^{\leq r}.$$

The upward-zeta transform of $\theta^{>r}$ vanishes on every nonempty level

$$1 \leq |S| \leq r.$$

Let

$$\Omega_n^\circ = U \subseteq [n] : U \neq [n].$$

For $U \in \Omega_n^\circ$, set

$$T = U^c.$$

Then $T \neq \varnothing$. Define

$$q(U) = (-1)^{|U|} \theta^{>r}_{U^c}.$$

Lemma 7.1. Complement-degree lemma The function q is the restriction to Ω_n° of a real multilinear polynomial of degree at most $n-r$. Proof Let $h^{>r}$ be the upward-zeta transform of $\theta^{>r}$. Möbius inversion gives

$$\theta^{>r}_T \sum_{S\supseteq T} (-1)^{|S|-|T|} h^{>r}_S.$$

Put

$$T = U^c \qquad \text{and} \qquad R = S^c.$$

Then $S \supseteq T$ is equivalent to $R \subseteq U$, and

$$|S| - |T|(n - |R|) - (n - |U|)|U| - |R|.$$

Therefore

$$q(U)(-1)^{|U|}\theta_{U^c}^{>r} \sum_{R \subseteq U} (-1)^{|R|} h_{R^c}^{>r}.$$

Since $h_S^{>r} = 0$ whenever $S \neq \varnothing$ and $|S| \leq r$, all nonzero terms satisfy

$$|R^c| > r.$$

Equivalently,

$$|R| < n - r.$$

Thus q is represented by a multilinear polynomial of degree at most $n - r - 1$, and hence certainly of degree at most $n - r$. $\square$

8 Projection separation from signed simplices

Let Ω be a finite set and let

$$V \subseteq \mathbb{R}^{\Omega}$$

be a linear subspace. Let

$$\xi \in \pm 1^{\Omega}.$$

Define the signed simplex

$$\mathcal{C}_{\xi} = \xi \cdot \mu : \mu \in \Delta(\Omega),$$

where

$$(\xi \cdot \mu)\omega = \xi\omega\mu_{\omega}.$$

Let Π_V be the orthogonal projection onto V with respect to the counting inner product

$$\langle f, g \rangle \sum_{\omega \in \Omega} f_{\omega} g_{\omega}.$$

Lemma 8.1. Projection separation If

$$|\Pi_V \xi|_{\infty} \leq \tau,$$

then for every $q \in V$ and every $\mu \in \Delta(\Omega)$,

$$|q - \xi \cdot \mu|_1 \geq \frac{1 - \tau}{1 + \tau}.$$

Proof Let

$$\delta = |q - \xi \cdot \mu|_1.$$

Since

$$\langle \xi, \xi \cdot \mu \rangle \sum_{\omega} \mu_\omega 1,$$

we have

$$\langle \xi, q \rangle 1 + \langle \xi, q - \xi \cdot \mu \rangle \geq 1 - \delta.$$

Also,

$$|q|_1 \leq |\xi \cdot \mu|_1 + \delta 1 + \delta.$$

Since $q \in V$,

$$\langle \xi, q \rangle \langle \Pi_V \xi, q \rangle \leq |\Pi_V \xi|_\infty |q|_1 \leq \tau(1 + \delta).$$

Combining the last two inequalities,

$$1 - \delta \leq \tau(1 + \delta),$$

and hence

$$\delta \geq \frac{1 - \tau}{1 + \tau}.$$

□

9 Random projection on the punctured cube

Let

$$\Omega_n = 0, 1^n, \qquad N = 2^n.$$

We identify Ω_n with $2^{[n]}$. Let

$$x_* = [n]$$

be the top point, and set

$$\Omega_n^\circ = \Omega_n \setminus x_*.$$

Let $L_{\leq d}$ be the space of real multilinear polynomials of degree at most d on the full cube, and let $W_{\leq d}$ be its restriction to Ω_n°. Put

$$M = \binom{n}{\leq d}.$$

Lemma 9.1. Punctured projection estimate Assume

$$d < n \qquad \text{and} \qquad M/N \to 0.$$

Let

$$\xi \in \pm 1^{\Omega_n^\circ}$$

be uniformly random. Let Π be the orthogonal projection onto $W_{\le d}$ in $\ell^2(\Omega_n^\circ)$. Then

$$|\Pi\xi|_\infty O_{\mathbb{P}}\left(\sqrt{\frac{Mn}{N}}\right).$$

In particular, if

$$M/N \le 2^{-\alpha n}$$

for some $\alpha > 0$, then

$$|\Pi\xi|_\infty = 2^{-\Omega(n)}$$

with high probability. Proof First, the restriction map

$$L_{\le d} \to W_{\le d}$$

is injective. Indeed, if $P \in L_{\le d}$ vanished on all points of the cube except possibly $x_* = [n]$, then, as a function on the full cube, it would be a scalar multiple of

$$\mathbf{1}_{[n]}(x) = x_1 x_2 \cdots x_n.$$

This function has multilinear degree n. Since $d < n$, such a polynomial P must be zero. Hence

$$\dim W_{\le d} = M.$$

On the full cube, normalized Walsh characters of degree at most d form an orthonormal basis of $L_{\le d}$. Hence the full projection matrix P onto $L_{\le d}$ has diagonal

$$P_{xx} = M/N$$

for every $x \in \Omega_n$. Let E be the $N \times M$ matrix with orthonormal columns spanning $L_{\le d}$. Let v be the row of E corresponding to the deleted point x_*. After deleting this row, the Gram matrix becomes

$$I - vv^{\mathsf{T}}.$$

Since

$$|v|_2^2 = P_{x_* x_*} = M/N < 1$$

for all large n, its inverse is

$$(I - vv^{\mathsf{T}})^{-1} I + \frac{vv^{\mathsf{T}}}{1 - |v|_2^2}.$$

Therefore, for $x \neq x_*$, the restricted projection matrix (P') satisfies

$$P'_{xx} = P_{xx} + \frac{P_{xx_*}^2}{1 - P_{x_*x_*}}.$$

Since P is positive semidefinite,

$$P_{xx_*}^2 \le P_{xx} P_{x_*x_*} \left(\frac{M}{N}\right)^2.$$

Therefore, for all large n,

$$P'_{xx} \le \frac{2M}{N}.$$

For fixed $x \in \Omega_n^\circ$,

$$(\Pi\xi)(x) = \sum_{y \in \Omega_n^\circ} P'_{xy} \xi_y.$$

Since (P') is an orthogonal projection,

$$\sum_y (P'_{xy})^2 = P'_{xx} \le \frac{2M}{N}.$$

Hoeffding's inequality gives

$$\Pr\left(|(\Pi\xi)(x)| > t\right) \le 2 \exp\left(-\frac{t^2}{4M/N}\right).$$

Taking

$$t = A\sqrt{\frac{Mn}{N}}$$

with A sufficiently large and union-bounding over at most N points proves the first claim. If $M/N \le 2^{-\alpha n}$, then this bound is exponentially small in n. $\square$

10 A general off-origin slack criterion

Let D_n, r_n be integer sequences satisfying

$$1 \le r_n \le \min(D_n, n)$$

eventually. Put

$$d_n = n - r_n,$$

$$M_n = \binom{n}{\le d_n}, \qquad N_n = 2^n,$$

and

$$\mathfrak{m}_n^* \mathfrak{m}^*(D_n, r_n) \max_{1 \le s \le r_n} \frac{2^s - 1}{\binom{D_n}{s}}.$$

Theorem 10.1. General off-origin slack criterion Assume: there exists $\varepsilon > 0$ such that

$$\frac{r_n}{n} > \frac{1}{2} + \varepsilon$$

for all sufficiently large n; $\mathfrak{m}_n^* \to 0$. Then, for uniformly random $G_n : 0, 1^n \to \pm 1$, with high probability,

$$\eta_{D_n}(G_n) \geq \frac{1 - \tau_n}{1 + \tau_n} \cdot \frac{1}{\mathfrak{m}_n^*} - 1,$$

where

$$\tau_n = O\left(\sqrt{\frac{M_n n}{N_n}}\right) 2^{-\Omega(n)}.$$

In particular, if

$$\frac{\tau_n}{\mathfrak{m}_n^*} = o(1),$$

then

$$\eta_{D_n}(G_n) \geq \frac{1}{\mathfrak{m}_n^*} - 1 - o(1).$$

Proof Normalize

$$G_\varnothing = 1$$

by multiplying G globally by $G_\varnothing$. The nonempty signs remain independent uniform signs. Let ν be any probability measure on nonempty subsets, and set

$$\theta_T = \nu_T G_T.$$

Let h be its upward-zeta transform. Let $\theta^{\leq r_n}$ and $\theta^{> r_n}$ be defined as above. By Lemma 6.1,

$$|\theta^{\leq r_n}|_1 \leq \mathfrak{m}_n^* B_{D_n,G}(\nu).$$

Now pass to complement variables. Define

$$q(U) = (-1)^{|U|} \theta_{U^c}^{> r_n}, \qquad U \neq [n].$$

By Lemma 7.1,

$$q \in W_{\leq d_n}.$$

Define

$$\xi_U = (-1)^{|U|} G_{U^c}, \qquad \mu_U = \nu_{U^c}.$$

Then ξ is a uniform random sign vector on Ω_n°, μ is a probability measure on Ω_n°, and

$$\xi \cdot \mu \in \mathcal{C}_\xi.$$

Moreover,

$$|q-\xi\cdot\mu|_1|\theta^{>r_n}-\theta|_1|\theta^{\le r_n}|_1\le\mathfrak{m}_n^*B_{D_n,G}(\nu).$$

By assumption 1,

$$d_n=n-r_n<\left(\frac{1}{2}-\varepsilon\right)n$$

for all sufficiently large n. Therefore

$$M_n/N_n\le 2^{-\alpha n}$$

for some $\alpha>0$. By Lemma 9.1, with high probability,

$$|\Pi_{W_{\le d_n}}\xi|_\infty\le\tau_n,$$

where

$$\tau_n=O\left(\sqrt{\frac{M_n n}{N_n}}\right)=2^{-\Omega(n)}.$$

On this high-probability event, Lemma 8.1 gives

$$\mathrm{dist}_{\ell^1}(W_{\le d_n},\mathcal{C}_\xi)\ge\frac{1-\tau_n}{1+\tau_n}.$$

Thus, uniformly for every ν,

$$\mathfrak{m}_n^*B_{D_n,G}(\nu)\ge\frac{1-\tau_n}{1+\tau_n}.$$

Equivalently,

$$B_{D_n,G}(\nu)\ge\frac{1-\tau_n}{1+\tau_n}\cdot\frac{1}{\mathfrak{m}_n^*}.$$

Since

$$A_G(\nu)\ge -1,$$

Proposition 5.1. yields

$$\eta_{D_n}(G_n)\min_\nu(A_G(\nu)+B_{D_n,G}(\nu))\ge -1+\frac{1-\tau_n}{1+\tau_n}\cdot\frac{1}{\mathfrak{m}_n^*}.$$

This proves the theorem. □ Corollary 10.2. Saturation criterion Under the assumptions of Theorem 10.1, if

$$\frac{1}{\mathfrak{m}_n^*}\to\infty,$$

then

$$\Pr(\gamma_{D_n}(G_n)=1)\to 1.$$

Proof Theorem 10.1 gives

$$\eta_{D_n}(G_n) \to \infty$$

with high probability. In particular,

$$\eta_{D_n}(G_n) \geq 1$$

with high probability. The empty point has margin exactly 1 by choosing

$$a_\varnothing = G_\varnothing.$$

Hence the full margin satisfies

$$\gamma_{D_n}(G_n) \geq 1.$$

Since always

$$\gamma_{D_n}(G_n) \leq 1,$$

we get

$$\gamma_{D_n}(G_n) = 1$$

with high probability. □

11 Linear-degree theorem and optimal off-origin slack

Let

$$H_2(t) = -t \log_2 t - (1-t) \log_2(1-t)$$

be the binary entropy function. Let $\rho_* \in (1/2, 1)$ be the unique solution of

$$H_2(\rho_*) = \rho_*.$$

Since $H_2(t) - t$ is strictly decreasing on ((1/2,1)), positive at $t = 1/2$, and negative at $t = 1$, this solution is unique. Numerically,

$$\rho_* = 0.7729078047\ldots, \qquad \frac{1}{2\rho_*} = 0.6469076867\ldots.$$

Theorem 11.1. Optimal off-origin slack in the linear regime Let D_n be an integer sequence satisfying

$$0 < \liminf_{n\to\infty} \frac{D_n}{n} \leq \limsup_{n\to\infty} \frac{D_n}{n} < \infty,$$

and

$$\liminf_{n\to\infty} \frac{D_n}{n} > \frac{1}{2\rho_*}.$$

Then, for uniformly random

$$G_n : 0, 1^n \to \pm 1,$$

one has, with high probability,

$$\eta_{D_n}(G_n) = D_n - 1 - o(1).$$

Consequently,

$$\Pr(\gamma_{D_n}(G_n) = 1) \to 1.$$

Proof Let

$$c_- = \liminf_{n\to\infty} \frac{D_n}{n}.$$

By hypothesis,

$$c_- > \frac{1}{2\rho_*}.$$

Choose c_0 such that

$$\frac{1}{2\rho_*} < c_0 < c_-.$$

Then, for all sufficiently large n,

$$D_n \geq c_0 n.$$

Choose β satisfying

$$\frac{1}{2} < \beta < \min(1, \rho_* c_0).$$

Set

$$r_n = \lfloor \beta n \rfloor.$$

Then

$$r_n \leq n$$

for all large n. Since $\rho_* < 1$,

$$\beta < \rho_* c_0 < c_0,$$

and therefore

$$r_n \leq D_n$$

for all large n. Also,

$$\frac{r_n}{n} \to \beta > \frac{1}{2}.$$

Choose ρ_0 such that

$$\frac{\beta}{c_0} < \rho_0 < \rho_*.$$

Then, for all sufficiently large n,

$$\frac{r_n}{D_n} \leq \rho_0.$$

We estimate

$$\mathfrak{m}^*(D_n, r_n) \max_{1 \leq s \leq r_n} \frac{2^s - 1}{\binom{D_n}{s}}.$$

At $s = 1$,

$$\frac{2^1 - 1}{\binom{D_n}{1}} \frac{1}{D_n}.$$

For $s \geq 2$, define

$$\alpha_s = \frac{2^s}{\binom{D_n}{s}}.$$

Then

$$\frac{2^s - 1}{\binom{D_n}{s}} \leq \alpha_s.$$

Moreover,

$$\frac{\alpha_{s+1}}{\alpha_s} \frac{2(s+1)}{D_n - s}.$$

This ratio is increasing in s. Hence the sequence α_s decreases and then increases, so on the interval $2 \leq s \leq r_n$ its maximum is attained at one of the endpoints $s = 2$ or $s = r_n$. At $s = 2$,

$$\alpha_2 \frac{4}{\binom{D_n}{2}} O(D_n^{-2}) < D_n^{-1}$$

for all sufficiently large n. It remains to control $s = r_n$. Put

$$t_n = \frac{r_n}{D_n}.$$

We have

$$t_n \leq \rho_0 < \rho_*.$$

Since $D_n = O(n)$ by hypothesis and $r_n \sim \beta n$, there is a constant $t_0 > 0$ such that

$$t_n \geq t_0$$

for all sufficiently large n. Therefore t_n lies in a compact subinterval of $(0, \rho_*)$. Hence there exists $\delta > 0$ such that

$$H_2(t_n) - t_n \geq \delta$$

for all sufficiently large n. By Stirling's formula, uniformly in this compact range,

$$\binom{D_n}{r_n} 2^{D_n H_2(t_n)+o(n)}.$$

Thus

$$\frac{2^{r_n}}{\binom{D_n}{r_n}} 2^{D_n t_n - D_n H_2(t_n)+o(n)} 2^{-D_n(H_2(t_n)-t_n)+o(n)} 2^{-\Omega(n)}.$$

Consequently,

$$\mathfrak{m}^*(D_n, r_n) = \frac{1}{D_n}$$

for all sufficiently large n. The hypotheses of Theorem 10.1 are satisfied. Moreover,

$$\tau_n = 2^{-\Omega(n)}.$$

Since $D_n = O(n)$,

$$D_n \tau_n = o(1).$$

Therefore Theorem 10.1 gives

$$\eta_{D_n}(G_n) \geq \frac{1-\tau_n}{1+\tau_n} D_n - 1D_n - 1 - o(1).$$

It remains to prove the matching upper bound. Normalize $G_\varnothing = 1$. The singleton signs

$$G_1, \ldots, G_n$$

are independent uniform signs. With probability $1-2^{-n}$, at least one singleton has sign -1. Choose such an i. For any feasible polynomial with

$$a_\varnothing = 1,$$

we have

$$p(1_i) = 1 + a_i, \qquad |a_i| \leq D_n.$$

Since $G_i = -1$, the margin at this point is

$$G_i p(1_i) - (1 + a_i) \leq D_n - 1.$$

Thus, with high probability,

$$\eta_{D_n}(G_n) \leq D_n - 1.$$

Combining the lower and upper bounds,

$$\eta_{D_n}(G_n) = D_n - 1 - o(1)$$

with high probability. Since $D_n \to \infty$, this implies

$$\eta_{D_n}(G_n) \geq 1$$

with high probability. Therefore

$$\gamma_{D_n}(G_n) = 1$$

with high probability. □

12 Homogeneous-form version

Theorem 11.1. has an equivalent homogeneous formulation. Let D_n be a linear degree sequence satisfying

$$0 < \liminf_{n\to\infty} \frac{D_n}{n} \le \limsup_{n\to\infty} \frac{D_n}{n} < \infty$$

and

$$\liminf_{n\to\infty} \frac{D_n}{n} > \frac{1}{2\rho_*}.$$

Then, with high probability over uniformly random G_n, there exists a homogeneous form

$$F \in \mathbb{R}[X_0, \ldots, X_n]_{D_n}$$

whose homogeneous coefficients all lie in ([-1,1]), such that

$$G_n(0, \ldots, 0)F(1, 0, \ldots, 0) = 1$$

and

$$G_n(x)F(1, x) \ge D_n - 1 - o(1)$$

for every nonzero

$$x \in 0, 1^n.$$

In particular,

$$G_n(x)F(1, x) > 0 \qquad \forall x \in 0, 1^n$$

with high probability. This follows directly from Theorem 11.1 and the exact realization of the coefficient box by homogeneous forms from Section 2.

13 A pseudorandom variant

The proof uses randomness only through the projection estimate

$$|\Pi_{W_{\le d}} \xi|_\infty = o(1).$$

This estimate also holds for sufficiently high-wise independent signs. Proposition 13.1. (O(n))-wise independence suffices for the projection step Fix $\varepsilon > 0$, and let

$$d \le \left(\frac{1}{2} - \varepsilon\right) n.$$

Let

$$\xi \in \pm 1^{\Omega_n^\circ}$$

be k-wise independent with

$$k \geq C_\varepsilon n$$

for a sufficiently large constant C_ε. Then

$$|\Pi_{W_{\leq d}} \xi|_\infty = o(1)$$

with high probability. Consequently, the saturation and off-origin slack conclusions above remain valid under any sign distribution for which, after normalizing the origin, the induced sign vector

$$\left((-1)^{|U|} G_{U^c}\right)_{U \in \Omega_n^\circ}$$

is (O(n))-wise independent. Proof Let (P') be the projection matrix onto $W_{\leq d}$. As in Lemma 9.1,

$$\sum_y (P'_{xy})^2 = P'xx \leq C_0 \frac{M}{N},$$

where

$$M = \binom{n}{\leq d}, \qquad N = 2^n.$$

Since

$$d \leq \left(\frac{1}{2} - \varepsilon\right) n,$$

there is $\alpha > 0$ such that

$$M/N \leq 2^{-\alpha n}.$$

For fixed x,

$$(\Pi \xi)(x) = \sum_y P'_{xy} \xi_y.$$

Because ξ is k-wise independent, its moments up to order k agree with those of fully independent signs. For even k, the Khintchine moment bound gives

$$\mathbb{E}|(\Pi \xi)(x)|^k \leq (C\sqrt{k})^k \left(\sum_y (P'_{xy})^2\right)^{k/2}.$$

Thus

$$\mathbb{E}|(\Pi \xi)(x)|^k \leq (C\sqrt{k})^k \left(C_0 \frac{M}{N}\right)^{k/2}.$$

Let

$$t = A\sqrt{\frac{Mn}{N}}.$$

By Markov's inequality,

$$\Pr(|(\Pi\xi)(x)| > t) \le \left(\frac{C\sqrt{k}}{A\sqrt{n}}\right)^k.$$

Taking

$$k = C_1 n$$

and choosing A, C_1 so that the right-hand side is at most 2^{-3n}, a union bound over at most 2^n points gives

$$\Pr(|\Pi\xi|_\infty > t) \le 2^{-2n}.$$

Since

$$t = A\sqrt{\frac{Mn}{N}} = o(1),$$

the result follows. □

14 Toward integer and ternary lifts

The results in this paper are over real coefficients. They do not assert the existence of integer or ternary homogeneous coefficient representations. Let

$$Z_{T,S} = \mathbf{1}_{S\subseteq T}$$

be the Boolean zeta matrix, restricted to columns $|S| \le D$. A rounding theorem converting real coefficients inside the box to integer coefficients while preserving signs would imply ternary homogeneous lifts: every integer coefficient

$$b_S \in [-\binom{D}{|S|}, \binom{D}{|S|}]$$

can be written as a sum of $\binom{D}{|S|}$ elements of $-1, 0, 1$, which can then be distributed among the homogeneous monomials reducing to x_S. The optimal off-origin slack

$$D - 1 - o(1)$$

shows that the real solution has large non-origin margin. However, this margin is only linear in D, and it is not by itself a rounding theorem. Universal, independent, or level-wise rounding estimates are not supplied by the present argument. A discrete version would require new correlated rounding methods or a direct lattice construction.

15 Scope and limitations

The main theorem is a real-coefficient result. It does not imply integer, ternary, or rounded homogeneous coefficient representations. The constant

$$\frac{1}{2\rho_*} = 0.6469076867\ldots$$

is a threshold of the present method, not claimed to be optimal. The proof requires a truncation level

$$r > n/2$$

so that the complementary polynomial space has exponentially small dimension relative to 2^n, and also requires

$$r/D < \rho_*$$

so that the sharp weighted-zeta localization parameter

$$\mathfrak{m}^*(D,r) \max_{1\le s\le r} \frac{2^s-1}{\binom{D}{s}}$$

is dominated by the singleton level $1/D$. These two inequalities combine to force

$$D > \frac{n}{2\rho_*}.$$

The theorem is stated in the linear regime $D_n = \Theta(n)$. This is the regime in which the method yields the asymptotically sharp off-origin margin

$$D_n - 1 - o(1).$$

For more rapidly growing D_n, the general criterion of Theorem 10.1 remains valid, but the additive error term must be tracked separately. Classical random threshold-degree results concern unbounded real coefficients. The present theorem concerns a specific bounded coefficient body arising from homogeneous height-one lifts. It is therefore a bounded-coefficient random representation theorem.

16 Summary of the argument

The proof may be summarized as follows. A homogeneous degree-D, height-one real lift induces exactly the multilinear box

$$|a_S| \le \binom{D}{|S|}.$$

The off-origin margin has the exact dual form

$$\eta_D(G) \min_{\nu} \big(A_G(\nu) + B_{D,G}(\nu)\big),$$

where $B_{D,G}$ is a weighted upward-zeta norm. For every dual measure ν, the low-level Möbius inverse of

$$\theta_T = \nu_T G_T$$

satisfies

$$|\theta^{\le r}|_1 \le \mathfrak{m}^*(D,r) B_{D,G}(\nu).$$

Removing this low-level part leaves high-zeta data. After complementing subsets and applying a parity sign, this high-zeta data becomes a low-degree polynomial of degree at most $n-r$. If

$$r > n/2 + \varepsilon n,$$

then the degree-((n-r)) polynomial space has exponentially small dimension relative to the cube. A random signed simplex is then ℓ^1-separated from that subspace. Therefore

$$\mathfrak{m}^*(D,r)B_{D,G}(\nu) \geq 1 - o(1)$$

uniformly in ν. In the linear regime $D = cn$, with

$$c > \frac{1}{2\rho_*},$$

one can choose $r > n/2$ with

$$r/D < \rho_*.$$

In this range,

$$\mathfrak{m}^*(D,r) = \frac{1}{D}.$$

Hence

$$B_{D,G}(\nu) \geq D - o(1)$$

uniformly in ν, and since

$$A_G(\nu) \geq -1,$$

one obtains

$$\eta_D(G) \geq D - 1 - o(1).$$

A matching upper bound $D-1$ holds with high probability because, after normalizing $G_\varnothing = 1$, at least one singleton has sign -1 with high probability. This proves

$$\eta_{D_n}(G_n) = D_n - 1 - o(1)$$

and therefore

$$\gamma_{D_n}(G_n) = 1$$

with high probability.

References

References

[1] N. Alon and Z. Füredi, Covering the cube by affine hyperplanes, European Journal of Combinatorics 14 1993, 79–83. [BCPS18] A. Bishnoi, P. L. Clark, A. Potukuchi, and J. R. Schmitt, On

zeros of a polynomial in a finite grid, Combinatorics, Probability and Computing 27 2018, 310–333. [BV19] P. Baldi and R. Vershynin, Polynomial threshold functions, hyperplane arrangements, and random tensors, SIAM Journal on Mathematics of Data Science 1 2019, 699–729. [Mur71] S. Muroga, Threshold Logic and Its Applications, Wiley-Interscience, 1971. [OD14] R. O'Donnell, Analysis of Boolean Functions, Cambridge University Press, 2014. [OS08] R. O'Donnell and R. A. Servedio, Extremal Properties of Polynomial Threshold Functions, Journal of Computer and System Sciences 74 2008, 298–312.

Declaration of generative AI and AI-assisted technologies in the writing process

During the preparation of this work, ChatGPT, by OpenAI, was used to assist with mathematical drafting, formalization, review, and editing. This work is shared as a preliminary AI-assisted mathematical note. The mathematical content may have been only partially reviewed and may contain errors; it should not be treated as peer-reviewed or as a fully verified manuscript.